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On Topological Field Theories and Duality

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Abstract

Topologically non trivial effects appearing in the discussion of duality transformations in higher genus manifolds are discussed in a simple example. Their relation with the properties of Topological Field Theories is established.

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Duality transformations [1] [2] [3] [4] [5] are constructed with the aim to relate different models of particles, strings and other extended objects by establishing equivalences between their spectrum and observables. For spin systems [2] and some two dimensional field theories [3] they have been constructed explicitly and present interesting features like strong coupling to weak coupling mappings or definite relations between solitons and Fock space states. However in most of the cases where their existence have been conjectured only partial evidence of the desired correspondences have been established, mainly in the form of particle spectrum identifications. In the path integral approach the search of duality transformations translate to that of an adequate equivalence between partition functions or generating functionals. There, space is also opened to apply related ideas to restricted low energy effective actions [4].

Dualized models have been obtained using path integrals introducing auxiliary fields in the path integral conveniently restricted and integrating out the original fields (or part of them)[5]. For 2-D and some 3-D models the results obtained by this method [6] have been shown to match those obtained in the operatorial approach [2] [3]. In this letter we show by taking a simple model that this procedure corresponds to a coupling with a topological field theory. This introduces the topological properties of the base manifold into the formalism, and gives a dynamical function to the topological field theories.

There are essentially two forms in which a least action principle implements a linear restriction of the form $G^{ij}\varphi_j = 0$: Introducing a quadratic lagrangian density $\mathcal{L} = \varphi_i G^{ij}\varphi_j$ or by means of a Lagrange multiplier. In the path integral approach the same effect of incorporating a factor of a power of $1/det(G)$ is obtained. Using the Lagrange multiplier one has as an intermediate step

$$I(\varphi) = \int \mathcal{D}\varphi \delta(G\varphi) exp - \int \mathcal{L}(\varphi) d^D x, \tag{1}$$

which allows for additional factors. This situation is somehow modified when the operator G applied to the fields is non-singular as occurs with gauge systems. In this case, care has to be taken with the zero eigenvalues of the the operator by means of some procedure which ultimately corresponds to the introduction of a modified measure. One has also the additional restriction of looking only to gauge invariant aspects of the model. This is the situation

one faces when one tries to impose the restriction $F_{\mu\nu}(A) = 0$, or more generally, $dA = 0$ on gauge fields. If no other fields are involved after taking care of the longitudinal sector in the path integral, the two options above correspond to nothing else but a Chern-Simons-like topological field theory [7] or a topological BF model of coupled antisymmetric and vector fields [8],[9]. The correct integration measure is best obtained by imposing $BRST$ invariance of the effective action. This leads to the complete definition of the corresponding topological field theories [7][8] [9]. In this sense and stressing the structure of (1) we note that BF theories are the adequate tools to define the restrictions $\delta(dA)$ or $\delta(F_{\mu\nu}(A))$ into the path integral framework. Going into the details let us write the partition function for such models [8] [9]

$$Z[0] = \int \mathcal{D}A\mathcal{D}B\mathcal{D}h e^{-\int(\mathcal{L}_{BF}+\mathcal{L}_{gf})d^D x}, \tag{2}$$

where $\mathcal{D}h$ stands for the integration on the complete set of ghost and auxiliary fields, $\mathcal{L}_{gf}$ is the gauge fixing term of the lagrangian density and $\mathcal{L}_{BF}$ is the BF lagrangian. This is given in the general case by $\mathcal{L}_{BF} = B \wedge F(A)$ with B a $(D-p-2)$-form, A an $p+1$-form and $F = dA$. In the particular case of $D = 3$ and A a vector field we have the simple expression

$$\mathcal{L}_{BF} = B_\mu F^\mu(A) = B_\mu \epsilon^{\mu\nu\rho}\partial_\nu A_\rho \tag{3}$$

which will be useful later. Here, $F_\mu(A) = 1/2\epsilon_{\mu\nu\rho}F^{\nu\rho}(A)$ and the conditions $F_\mu(A) = 0$ and $F_{\mu\nu}(A) = 0$ are completely equivalent since $F_0 = F_{12}$, $F_1 = F_{20}$ and $F_2 = -F_{10}$.

In recent works [5], the restriction of zero curvature imposed to an auxiliary gauge field has been used as a fundamental ingredient for the introduction of dual variables and dualized models in the path integral approach. The essential steps of this method are the following. First, a gauge symmetry is identified and the corresponding gauge model is considered restricted to the condition $F_{\mu\nu}(A) = 0$. This is implemented by means of a Lagrange multiplier. In fact, as we show below in a concrete example the symmetry considered may be one of only some terms of the lagrangian density and the main line of reasoning remains untouched. Second, after some intermediate manipulations which depend on the specific model considered, the auxiliary field or the lagrange multiplier becomes the fundamental variable of the dualized model. The appearance or not of a mapping between the strong coupling

and the weak coupling of the models is not granted by this procedure and depends of the systems under consideration.

Since topological field theories are distinguished for being able to extract the topological non-trivial information of the manifolds where they are formulated, stressing their role in the construction of the dualized models appears as a promising way of incorporating this issues in the formulation. In what follows we will show how global aspects intervene the implementation of duality in the rather simple but non-trivial example of vector models in 3-D.

In 3-D massive, parity odd excitations may be described by three different vector models [10] which are respectively the topologically massive model (TMM), the so-called self-dual model (SDM) (here self-dual is not related with duality as we are interested but refers to a property of the equations of motion of the model) and a third model which we will call the intermediate model (IM). The corresponding lagrangian densities are given by

$$\mathcal{L}_{TMM} = -\frac{m}{2}(\epsilon^{\mu\nu\rho}\partial_\nu A_\rho)(\epsilon^{\mu\alpha\beta}\partial_\alpha A_\beta) + \frac{1}{2}A_\mu\epsilon^{\mu\nu\rho}\partial_\nu A_\rho \tag{4}$$

$$\mathcal{L}_{SDM} = \frac{m}{2}a_\mu a^\mu - \frac{1}{2}a_\mu\epsilon^{\mu\nu\rho}\partial_\nu a_\rho \tag{5}$$

$$\mathcal{L}_{IM} = \frac{m}{2}a_\mu a^\mu - a_\mu\epsilon^{\mu\nu\rho}\partial_\nu A_\rho + \frac{1}{2}A_\mu\epsilon^{\mu\nu\rho}\partial_\nu A_\rho \tag{6}$$

These systems have been studied extensively from various points of view and may be shown to be locally equivalent by means of different analysis. Deser and Jackiw [10] provided the original proof of the equivalence between the (TMM) and the (SDM) solving the canonical equal-time algebra of the quantized fields in terms of a canonical free massive field. They also introduced the intermediate model as a master first order formulation of the other two: Taking variations respect to a_μ or A_μ in (6) and substituting back the resulting equation in $\mathcal{L}_{IM}$, one recovers respectively $\mathcal{L}_{TMM}$ or $\mathcal{L}_{SDM}$. The local equivalence of this models has also been discussed in the canonical Hamiltonian approach [11] and in fact it has been shown that the SDM, which is not a gauge theory, emerges as a gauge fixed version of the TMM in topologically trivial manifolds. On the other hand in higher genus manifolds, the TMM and the SDM are not equivalent. This is most easily established noting that the only solution of the SDM which satisfies $F_\mu(a) = 0$ is $a_\mu = 0$ in contrast to the TMM for which every flat connection is a solution [12].

Let us turn to the point we want to raise and note that the TMM may also be obtained as the dualized version of SDM when one applies the duality transformation described above. For genus zero manifolds, where the two systems are equivalent this provides just another way to show this equivalence. For higher genus manifolds as discussed above the systems are not globally equivalent and so we have a concrete example for which the duality transformation induces non trivial topological properties in the resulting model. To see this let us consider the partition function of the SDM in a genus zero manifold

$$Z_{SDM}[0] = \mathcal{N} \int \mathcal{D}a_\mu exp - \int \Big(\frac{m}{2} a_\mu a^\mu - \frac{1}{2} a_\mu \epsilon^{\mu\nu\rho} \partial_\nu a_\rho)\Big) d^3x. \tag{7}$$

For notational simplicity we write our equations for a locally flat metric but they generalize to the curved case. Next observe that the second term in $\mathcal{L}_{SDM}$ is invariant under the addition of a gradient. In genus zero manifolds if one introduces an auxiliary gauge fieldA_μ coupled to a_μ in the form

$$\mathcal{L}_{int}(a, A) = -\frac{1}{2}(a_\mu + A_\mu)\epsilon^{\mu\nu\rho}\partial_\nu(a_\rho + A_\rho) \tag{8}$$

and impose

$$F_\mu(A) = 0, \tag{9}$$

in order to perform the duality transformation, then, $A_\mu = \partial_\mu \Lambda$ is a pure gauge and $\mathcal{L}_{int}(a, A)$ is in fact equal to the second term of $\mathcal{L}_{SDM}$. In higher genus manifolds this is not true because there are solutions to (9) which cannot be written globally as pure gauges and is here that the non-trivial topological properties of the system find their way into the formulation. After introducing in such a way the dual variables we have in an arbitrary manifold,

$$Z^{Dual}_{SDM}[0] = \mathcal{N} \int \mathcal{D}A_\mu \mathcal{D}a_\mu \delta(F_\mu(A)) exp - \int \Big(\frac{m}{2} a_\mu a^\mu$$
$$-\frac{1}{2}(a_\mu + A_\mu)\epsilon^{\mu\nu\rho}\partial_\nu(a_\rho + A_\rho) + \text{gauge fixing terms}\Big) d^3x. \tag{10}$$

We introduce a Lagrange multiplier B_μ to promote the $\delta(F_\mu(A))$ to the lagrangian. To maintain our argument simple, we do not enter into the details of the gauge fixing procedure, which are well understood [8] [9] and amount to a proper definition of $\delta(F_\mu(A))$ and simply raise to the effective lagrangian

a gauge fixing term for each auxiliary field. To facilitate the Gaussian integration that follows, we choose the conditions $\partial^\mu(A_\mu + a_\mu - B_\mu) = 0$ for the A field and $\partial^\mu B_\mu = 0$ for the B field which are clearly allowed. We then have,

$$
\begin{aligned}
Z_{SDM}^{Dual}[0] = \mathcal{N} \int \mathcal{D}a_\mu \mathcal{D}B_\mu \mathcal{D}A_\mu exp - \int \Big(-\frac{1}{2}(a_\mu + A_\mu)\epsilon^{\mu\nu\rho}\partial_\nu(a_\rho + A_\rho) \\
+\frac{m}{2}a_\mu a^\mu + B_\mu(\epsilon^{\mu\nu\rho}\partial_\nu A_\rho) + \frac{1}{2\chi}(\partial_\mu B^\mu)(\partial_\nu B^\nu) \\
+\frac{1}{2\xi}\partial^\mu(A_\mu + a_\mu - B_\mu)\partial^\nu(A_\nu + a_\nu - B_\nu)\Big) d^3x. \quad (11)
\end{aligned}
$$

What we have obtained in this intermediate step is the partition function of a BF topological field theory coupled to a matter field described by the SDM. Now, one can perform the regular Gaussian integration in the field $\tilde{A}_\mu = A_\mu + a_\mu - B_\mu$, and we get

$$
\begin{aligned}
Z_{SDM}^{Dual}[0] = \tilde{\mathcal{N}} \int \mathcal{D}a_\mu \mathcal{D}B_\nu exp - \int \Big(\frac{m}{2}a_\mu a^\mu - a_\mu \epsilon^{\mu\nu\rho}\partial_\nu B_\rho \\
+\frac{1}{2}B_\mu \epsilon^{\mu\nu\rho}\partial_\nu B_\rho + \frac{1}{2\chi}(\partial^\mu B_\mu)(\partial^\nu B_\nu)\Big) d^3x = Z_{IM}[0]. \quad (12)
\end{aligned}
$$

This is the partition function of the IM, which may be also recognized as a Chern-Simons topological model coupled to the SDM. By simply performing the Gaussian integration in a_μ, we obtain directly as it was advanced the partition function of the TMM. For the reasons mentioned above, the dualized model we end up is not globally equivalent to the one we started with. This can be shown most clearly at this point by factorizing in the partition function the term which encodes the topological information of the manifold. To this end we take advantage of the gauge invariance of the system and proceed in the following way. Instead of integrate a_μ in (12), we can make the shift

$$
B_\mu \rightarrow B_\mu + a_\mu \quad (13)
$$

This leaves us with

$$
\begin{aligned}
Z_{SDM}^{Dual}[0] = \tilde{\mathcal{N}} \int \mathcal{D}a_\mu \mathcal{D}B_\nu exp - \int \Big(\frac{m}{2}a_\mu a^\mu - \frac{1}{2}a_\mu \epsilon^{\mu\nu\rho}\partial_\nu a_\rho \\
+\frac{1}{2}B_\mu \epsilon^{\mu\nu\rho}\partial_\nu B_\rho + \frac{1}{2\chi}\partial^\mu(B_\mu + a_\mu)\partial^\nu(B_\nu + a_\nu)\Big) d^3x. \quad (14)
\end{aligned}
$$

After recognizing that $\partial^\mu(B_\mu + a_\mu) = 0$ is an acceptable gauge fixing condition for the gauge field B_μ we end up with the factorized relation

$$Z_{SDM}^{Dual} = Z_{CS} Z_{SDM} \tag{15}$$

where Z_{CS} is the partition function of the Chern Simons topological theory. This is confirmed by a detailed computation in the Hamiltonian approach[12]. Some of this considerations generalize also to non-Abelian and tensor fields [13] [14].

A similar relation between the generating functionals of the models may also be obtained such that if we introduce the external current minimally coupled to the TMM (which is a gauge model and calls for it) we do not get this current minimally coupled to the SDM. This, together with the topological blindness of the SDM is relevant for the discussion of anyons within these models. [15].

Let us conclude by summarizing the most salient lessons we take from this analysis:

- Duality transformations are implemented by coupling the original model with a BF topological theory. In genus zero manifolds this do not introduce any difference but in higher genus manifolds the global equivalence of the models isno longer valid. In the case discussed above the net effect is a coupling of the matter fields with a Chern-Simons topological theory. This feature is likely to be generalized to other contexts and furnishes a dynamical function for the topological fields, theories as mentioned at the beginning. We note that this matches with the fact that although BF fields interacting with classical sources do not act with a force on them, they select the allowable trajectories on topological grounds [16].

- Our experience with the TMM and the SDM suggests also to look to models connected by a duality transformation as related by a gauge fixing procedure [11]. We note that the physical observables in the TMM are only the gauge invariant operators and this does not occur in the SDM for which other operators are also allowed as observables.

The issues discussed in this letter do not address the interesting possibility of duality between the particles of the TMM and the SDM and the

soliton spectrum of this or related models. On the other hand, most of the discussion presented here translate to more general contexts where duality transformations have been implemented in the functional approach. The conclusions derived from this minimal model should shed light to these more general cases. In particular one understands in a simple way why the dualized models should become sensible to the topological properties of the base manifold.

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